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A Fixed-Point Equation for the High-Temperature Phase of Discrete Lattice Spin Systems

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A fixed-point equation on an infinite-dimensional space is proposed as an alternative to the usual definition of the infinite-volume limit in discrete lattice spin systems in the high-temperature phase. It is argued heuristically that the free energy and correlation functions one obtains by solving this equation agree with the usual definitions of these quantities. A theorem is then proved that says that if a certain finite-volume condition is satisfied, then this fixed-point equation has a solution and the resulting free energy is analytic in the parameters in the Hamiltonian. For particular values of the temperature this finite-volume condition may be checked with the help of a computer. The two-dimensional Ising model is considered as a test case, and it is shown that the finite-volume condition is satisfied for $\beta \leqslant 0.77\beta_{\text{critical}}$.

KEY WORDS: Finite volume condition; high temperature phase; lattice spin system.

1. INTRODUCTION

Given a lattice spin system at sufficiently high temperature, there are various methods, e.g., high-temperature expansions or the Dobrushin uniqueness theorem, for proving it has all the properties one would expect. Suppose, however, that we are given a lattice spin system at some temperature which we suspect is above the critical temperature, but which is not sufficiently high that the usual methods apply. Is there some finite-volume calculation one can do that would prove that the system is in the high-temperature phase? One would expect that the amount of calculation required to show that the system is in the high-temperature phase would increase as one approaches the critical temperature. Consequently, such a method will never yields results valid in the entire high-temperature phase, only some subset of it. Thus, one should ask why a method that works in

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90% of the high-temperature region is of any more interest than a method that works in only 25% of the high-temperature region. The answer is that there are important problems that can be cast in the form of showing that a particular statistical mechanical system is in the high-temperature phase even though the usual methods do not apply. (By the phrase "high-temperature phase" I mean any phase in which there is a unique Gibbs state and exponentially decaying correlations. A system can be in such a phase because the temperature is high, but it can also be in such a phase when the temperature is not high.)

The best known example of such a problem is the work of Dobrushin *et al.*[6] on the Ising antiferromagnet in an external magnetic field. Dobrushin and Shlosman[5] have developed a sequence of finite-volume conditions, the verification of any one of which implies that the system is in the high-temperature phase. For the Ising antiferromagnet in a magnetic field the question of whether or not there is a reentrant phase transition as one increases the temperature can be reduced to showing that a particular statistical mechanical system is in the high-temperature phase. They were able to verify[6] one of the Dobrushin–Shlosman finite-volume conditions for this system with the help of a computer and thereby show that there is not a reentrant phase transition for the Ising antiferromagnet.

An important unsolved problem in this category is the majority rule renormalization group approach to critical phenomena in classical lattice spin systems.[12] In this implementation of the renormalization group for Ising-like models, the value of the block spin is defined to be the sign of the sum of the spins in the block. Hence the renormalization group map sends a Hamiltonian for Ising spins (i.e., spins which can only be ± 1) into another Hamiltonian for Ising spins. The numerical advantages of such a scheme are obvious, and indeed this map has been studied extensively numerically. The only rigorous work on this map has concerned regions of the parameter space far from the critical point. It is not even known that the map is well defined when the starting system is near the critical point. In fact, there is strong evidence that the map is not well defined for all Hamiltonians.[4,7–10]

The precise definition of the map is as follows. Let $H(\sigma)$ be some Hamiltonian for Ising-like spins. Divide the lattice into 3 by 3 blocks (to take a particular implementation of the map) and associate a block spin which also takes the values ± 1 to each block. Fix a configuration of these block spins and sum over all the original spins subject to the constraint that in each block the majority of the original spins agrees with the block spin. This constrained partition function will be a function of the block spins, so we write it as $\exp[\bar{H}(\bar{\sigma})]$, where $\bar{\sigma}$ denotes the block spins. This map from H to $\bar{H}$ is the renormalization group map. Of course, the above procedure requires taking an infinite-volume limit, and so this map is not *a priori* well defined. The folklore of the renormalization group is that even if the starting Hamiltonian H is precisely at the critical point, if we fix a choice of block spins $\bar{\sigma}$ and then consider the spin system in which we sum only over those spin configurations that satisfy the majority rule, then this constrained spin system should be in a high-temperature phase. In other words, introducing the block spins should make the correlation length finite. The critical nature of the starting Hamiltonian will manifest itself as one iterates the map. Indeed, if the starting Hamiltonian is critical, then under iteration it should be driven to a fixed point for the renormalization group map. Methods that could handle systems which are in the high-temperature phase but outside the region where the usual high-temperature methods apply are precisely what is needed to establish the existence of this map.

Another example of such a problem concerns the ground states of a certain class of quantum spin systems.[1] In these quantum spin systems the ground state can be written down explicitly in terms of valence bonds, but it is not easy to determine the properties of the ground state. Arovas *et al.*[3] showed that these quantum ground states have a representation in terms of a classical system at nonzero temperature. Showing that the ground state is disordered, which is believed to be the case for some of the models, then amounts to showing that this classical system is in the high-temperature phase. For one particular model, the spin-3/2 model on the hexagonal lattice, Kennedy *et al.*[11] were able to do this using the usual polymer expansion and a computer. However, the other models which we expect to have disordered ground states are outside the region where the convergence of the polymer expansion can be proved.

As mentioned earlier, Dobrushin and Shlosman have developed a sequence of finite-volume conditions, any one of which implies that the model is in the high-temperature phase. Moreover, for most discrete spin systems if the temperature is above the critical temperature, then their finite-volume condition should be satisfied for sufficiently large volumes. From a theoretical point of view this solves the problem posed at the start of the introduction. However, from a practical point of view it should be emphasized that if the method requires that one do a computation which involves all the spin configurations in the finite volume, then since the number of these spin configurations grows with the length L of the volume as q^{L^d} (d is the number of dimensions, q is the number of spin states), any such method will be restricted to a small volume and hence to systems with a fairly short correlation length. In our method the number of dimensions is effectively reduced by one, which greatly increases the feasibility of the required finite-volume calculations. As a test case, we have carried out computer calculations on a Sun workstation for the two-dimensional Ising model. We were able to verify our finite-volume condition for $\beta \leqslant 0.77\beta_{\text{critical}}$.

For a general class of ferromagnetic models, Aizenman[2] has shown that there is a sequence of finite-volume calculations which yield upper and lower bounds on the critical temperature which converge to the critical temperature as the volume increases. A sequence of convergent upper bounds had been found earlier by Simon.[13] While these results are quite strong, they do not apply to arbitrary discrete spin systems.

The heart of our method is to abandon the usual definitions of the infinite-volume limit and replace them with a fixed-point equation on an infinite-dimensional space. In Section 2 we will give a heuristic derivation of this fixed-point equation. We do not prove that this fixed-point equation is equivalent to the usual definitions of the infinite-volume limit. For very high temperatures it should be possible to use high-temperature expansions to prove this by putting the arguments of Section 2 on a rigorous footing. In Section 3 we prove a theorem which says that if a certain finite-volume condition is satisfied, then the fixed-point equation has a solution and the resulting free energy is analytic in the parameters in the Hamiltonian. This finite-volume condition is then checked numerically for several values of β. We do not prove that the solution is unique, but see the remark following the proof of Theorem 3.4. In Section 4 we derive a better fixed-point equation and show that there is also a finite-volume condition that implies that it has a solution. In Section 5 we show how to extract the correlation functions from our fixed-point equation and show that they, too, are analytic in the parameters in the Hamiltonian.

The remainder of the paper deals exclusively with the two-dimensional Ising model, so we should comment on the extent to which the methods developed here apply to some of the problems we have discussed. We will make extensive use of the fact that for Ising spins any function of the spins σ_i in a finite volume V can be written in a unique way as $\sum_W c(W)\,\sigma(W)$, where W is summed over all subsets of V, $\sigma(W)$ denotes $\prod_{i \in W} \sigma_i$, and the $c(W)$ are constants. This can be thought of as a Fourier transform on $Z/2$, and easily generalizes to any discrete spin system. The generalization from two to higher dimensions is also straightforward. Thus, our restriction to the two-dimensional Ising model rather than a general discrete spin system in an arbitrary number of dimensions is for purely pedagogical reasons.

It is not obvious how to apply the methods of this paper to the majority rule transformation. It can be done, and we have done some preliminary numerical work which reproduces the usual picture of the renormalization group flow. Unfortunately, we have no rigorous results for temperatures close to the critical temperature. (It should be noted that for the renormalization group transformation our method does not completely reduce the number of dimensions by one.)

For the special quantum spin systems whose ground states have a classical representation we need a method which applies to polymer systems. We do not know yet how to use the ideas here to develop such a method. For a general quantum spin system one can try to write it as a classical system in one higher dimension using a Feyman–Kac formula. This extra dimension will be continuous rather than discrete and so the methods presented here will require further development before they can be applied to such systems. The resulting classical system will in general have complex parameters in the Hamiltonian. Unlike probabilistic methods, our methods work for such Hamiltonians.

Finally, since we do not prove that our definition of the infinite-volume limit agrees with the usual one, we should comment on exactly what our method proves for a discrete spin model with the infinite-volume limit defined in the usual manner. The free energy that we define will be shown to be an analytic function of β in some region containing $\beta = 0$. If one can indeed use high-temperature expansions to prove that our definition of the free energy agrees with the usual definition at high enough temperatures, then we will have defined an analytic function which agrees with the usual free energy in a neighborhood of $\beta = 0$. This would then imply that either there is a temperature at which the usual free energy is not analytic, but none of its derivatives blow up, or the usual free energy is analytic in a region at least as large as the region of analyticity of our free energy. The same remarks apply to the correlation functions.

2. THE FIXED-POINT EQUATION

In this section we will derive the fixed-point equation in a heuristic fashion. The arguments in this section are not meant to be rigorous. Our purpose is to convince the reader at a heuristic level that the thermodynamic quantities, e.g., the free energy, that we obtain by solving the fixed-point equation are the same as those obtained from the usual definitions.

Consider the finite volume shown in Fig. 1. If we include in the Hamiltonian the terms that couple the interior of this region to the exterior, then the free energy for this region will be a function of the boundary spins. Any function of these spins can be written as a linear combination of the functions $\sigma(V)$, where $\sigma(V) = \prod_{i \in V} \sigma_i$. Thus,

$$\ln(Z) = \sum_{V} c(V)\, \sigma(V) \tag{2.1}$$

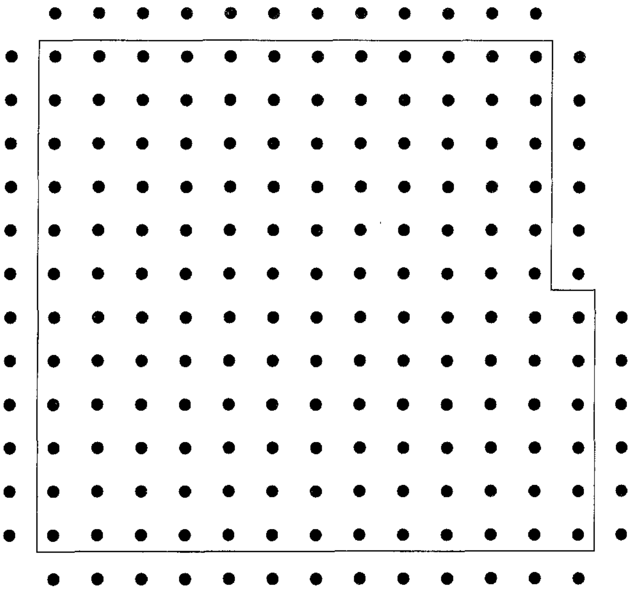

Fig. 1. The finite volume used in the heuristic derivation of the fixed-point equation.

where V is summed over all subsets of the set of boundary sites. If the temperature is above the critical temperature, then the coefficients $c(V)$ should decay as V grows. We assume that the volume in Fig. 1 is large and restrict our attention to the terms in (2.1) that are localized near the kink in the boundary. In Fig. 2 we have redrawn the kink and explicitly labeled the boundary spins. The free energy does not depend on σ_{-1}, but we have included this spin for later use.

Now consider what happens when we sum over the spin σ_0. There are two terms in the Hamiltonian that involve σ_0 and do not involve any of the spins which have already been summed out, namely, $\beta\sigma_0\sigma_1$ and $\beta\sigma_0\sigma_{-1}$. Define $\bar{c}(W)$ by

$$\sum_{\sigma_0} \exp\left[\sum_V c(V)\,\sigma(V) + \beta\sigma_0\sigma_1 + \beta\sigma_0\sigma_{-1}\right] = \exp\left[\sum_W \bar{c}(W)\,\sigma(W)\right] \tag{2.2}$$

In this equation V is summed over all finite subsets which do not contain -1, W is summed over all finite subsets which do not contain 0. If our volume is very large so that the distance from the kink to the top and bottom of the volume is large, then the free energy we obtain by summing over σ_0 should be essentially the same function of the boundary spins near the kink as the free energy before summing over σ_0; we need only relabel the spins appropriately. Thus, $\bar{c}(W)$ should be equal to $c(V)$ if V equals W

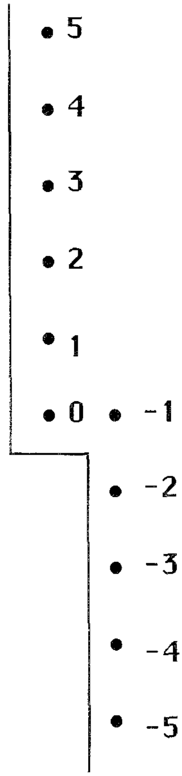

Fig. 2. An enlarged view of the kink in the boundary of the finite volume of Fig. 1. Only the boundary sites are shown. The spin at site 0 will be summed over next.

shifted by one lattice spacing. For example, $\bar{c}(1, 2)$ should be equal to $c(0, 1)$. In general, we should have

$$\bar{c}(W+1) = c(W) \tag{2.3}$$

where $W+1 = \{i+1 : i \in W\}$. This equality should be exact in the infinite-volume limit.

In Eq. (2.2) there are many terms $c(V)\,\sigma(V)$ which do not involve σ_0. If we define $f(W)$ by

$$\begin{aligned}\sum_{\sigma_0} \exp\left[\sum_{V:\,0\in V} c(V)\,\sigma(V) + \beta\sigma_0\sigma_1 + \beta\sigma_0\sigma_{-1}\right]& \\ = \exp\left[\sum_W f(W)\,\sigma(W)\right]&\end{aligned} \tag{2.4}$$

then $\bar{c}(W)$ is just $c(W)+f(W)$. If $-1 \in W$, then there is no $c(W)$ in Eq. (2.2), so $\bar{c}(W)$ just equals $f(W)$. To summarize,

$$\bar{c}(W)=\begin{cases} c(W)+f(W), & -1 \notin W \\ f(W), & -1 \in W \end{cases} \tag{2.5}$$

Combining (2.3) and (2.5), we have

$$c(W)=\begin{cases} c(W+1)+f(W+1), & -2 \notin W \\ f(W+1), & -2 \in W \end{cases} \tag{2.6}$$

In this equation W can be any finite subset which does not contain -1.

Thus far all we have done is formulate a transfer matrix approach. Equation (2.6) is already a fixed-point equation for the $c(V)$'s. We will improve this equation by eliminating all the $c(V)$'s with $0 \notin V$. We proceed by examples.

For $W=\{0, 1\}$ and its translates $\{1, 2\}$, $\{2, 3\}$,..., Eq. (2.6) yields

$$c(0, 1)=c(1, 2)+f(1, 2)$$

$$c(1, 2)=c(2, 3)+f(2, 3)$$

$$\cdots$$

Summing these equations, we obtain

$$c(0, 1)=\sum_{t=1}^{\infty} f(t, t+1)+c(\infty, \infty+1) \tag{2.7}$$

where $c(\infty, \infty+1)$ is short-hand for the limit as $t \to \infty$ of $c(t, t+1)$.

If we apply Eq. (2.6) to the sets $\{-1, -2\}$, $\{-2, -3\}$,..., we obtain

$$c(-3, -2)=f(-2, -1)$$

$$c(-4, -3)=c(-3, -2)+f(-3, -2)$$

$$c(-5, -4)=c(-4, -3)+f(-4, -3)$$

$$\cdots$$

These equations give

$$c(-\infty, -\infty+1)=\sum_{t=-\infty}^{-2} f(t, t+1) \tag{2.8}$$

As we move away from the kink the two pieces of the right boundary look the same. Hence $c(\infty, \infty+1)=c(-\infty, -\infty+1)$. Combining this equation with (2.7) and (2.8) yields

$$c(0, 1)=\sum_{t \neq 0, -1} f(t, t+1) \tag{2.9}$$

Since $f(W)$ only depends on the $c(V)$ with $0 \in V$, we have succeeded in obtaining an equation for $c(0, 1)$ which depends only on the $c(V)$ with $0 \in V$.

For any W which only contains nonnegative sites the argument above yields

$$c(W) = \sum_{t:W} f(W + t) \tag{2.10}$$

$W + t$ denotes the set $\{i + t: i \in W\}$ and t: W means that t is summed from $-\infty$ to ∞ except for the values $-k, -k + 1, \ldots, -1, 0$, where k is the largest element of W. Note that the t over which we sum can be characterized by the following two conditions:

(i) $0 \notin (W + t)$.

(ii) $W + t$ has the same geometric shape as W.

For sets W which contain at least one negative site the resulting equations are slightly different. In this case we obtain a finite set of equations. For example, consider $\{0, -4\}$. Equation (2.6) yields

$$c(0, -4) = c(1, -3) + f(1, -3)$$
$$c(1, -3) = c(2, -2) + f(2, -2)$$
$$c(2, -2) = f(3, -1)$$

Thus

$$c(0, -4) = f(1, -3) + f(2, -2) + f(3, -1)$$

Note that this equation is still given by Eq. (2.10) with t: W defined by conditions (i) and (ii) above. The same argument shows that Eq. (2.10) applies for any W which contains a negative site. Thus, Eq. (2.10) holds for any W.

Equation (2.10) is a fixed-point equation for the coefficients $\{c(V): 0 \in V, -1 \notin V\}$. We denote this set of coefficients by c. The functions $f(W)$ depend on c and on β. When we need to make this dependence explicit, we will write $f(W, c, \beta)$. Now define

$$F(W, c, \beta) = \sum_{t:\, W} f(W + t, c, \beta) \tag{2.11}$$

Then the fixed-point equation for c is

$$F(W, c, \beta) = c(W) \tag{2.12}$$

This equation must hold for all W which contain 0 and do not contain -1.

The solution of this equation will depend on β, and when we need to make this dependence explicit, we will write $c(\beta)$.

How do we obtain the free energy per site in the infinite-volume limit from the fixed-point equation? Consider Eq. (2.4). In the right side the sum over W includes the term where W is the empty set, i.e., $f(\varnothing)$. This term is the free energy per site. Thus the free energy per site equals $f(\varnothing, c(\beta), \beta)$, where $c(\beta)$ is the solution of (2.12).

3. EXISTENCE OF A SOLUTION

In this section we will prove that Eq. (2.12) has a fixed point if a certain finite-volume condition is satisfied. We must first give a rigorous definition of $F(c)$. If $g(V)$ is a real-valued function on the *finite* subsets V of $\{..., -2, -1, 0, 1, 2,...\}$ such that $\sum_V |g(V)| < \infty$, then $\sum_V g(V)\,\sigma(V)$ defines a function $g(\sigma)$. We define

$$\|g\| = \sum_V |g(V)| \tag{3.1}$$

We will only allow c's with finite $\|c\|$ in Eq. (2.12). The c's that occur in Eq. (2.12) have an additional property: $c(V) \neq 0$ only for V such that $0 \in V$. The Banach space in which we look for a solution to the fixed-point equation is the set of c's with this property and finite $\|c\|$. We will show that $F(c)$ and its Jacobian $DF(c)$ are defined and continuous on an open subset of this Banach space. We will use the norm (2.12) both for c's in this Banach space and for g's which may have $g(V) \neq 0$ for finite subsets V that do not contain 0.

Equation (2.4) can be formally rewritten as

$$f(W) = \frac{1}{N} \sum_\sigma \sigma(W) \ln \left\{ \sum_{\sigma_0} \exp \left[\sum_{V:\,0 \in V,\, -1 \notin V} c(V)\,\sigma(V) + \beta\sigma_0\sigma_1 + \beta\sigma_0\sigma_{-1} \right] \right\} \tag{3.2}$$

The sum over σ is over all spin configurations σ on the sites $\{..., -2, -1, 1, 2, 3,...\}$, and the $1/N$ normalizes this sum. Of course, this sum is infinite, so the above equation is not yet a rigorous definition of $f(W)$. We will say that c has finite support if there is a finite set S such that $c(V) \neq 0$ implies that $V \subset S$. For c with finite support, Eq. (3.2) is well defined, since we can replace the sum over σ by the sum over just the spin configurations on the set S. Hence $F(c)$ is well defined for c with finite support. We will initially work only with c's with finite support. Then we will extend our definitions and bounds to an open subset in the space of c's.

Let $DF(c)$ denote the Jacobian of F at c. Its matrix elements are

$\partial F(V)/\partial c(W)$, $0 \in V$, $0 \in W$. Since we are using the l^1 norm, the operator norm of $DF(c)$ is easily bounded by

$$\|DF(c)\| \leqslant \sup_{W:\, 0 \in W} \sum_{V:\, 0 \in V} \left| \frac{\partial F(V)}{\partial c(W)} \right| \tag{3.3}$$

From (2.11) we have

$$\frac{\partial F(V)}{\partial c(W)} = \sum_{t:\, V} \frac{\partial f(V+t)}{\partial c(W)}$$

For a function $g(\sigma)$ on spin configurations σ, define

$$\langle g \rangle = Z^{-1} \sum_{\sigma_0} g(\sigma) \exp\left[\sum_{V:\, 0 \in V,\, -1 \notin V} c(V)\, \sigma(V) + \beta \sigma_0 \sigma_1 + \beta \sigma_0 \sigma_{-1} \right] \tag{3.4}$$

where Z is defined by $\langle 1 \rangle = 1$. This expectation depends on c. When we need to make this dependence explicit, we will write $\langle \cdot \rangle_c$. For the moment $\langle g \rangle_c$ is only defined for c with finite support. Equation (3.2) now implies

$$\frac{\partial f(V+t)}{\partial c(W)} = \frac{1}{N} \sum_{\sigma} \sigma(V+t) \langle \sigma(W) \rangle$$

The quantity $\langle \sigma_0 \rangle$ is a function of the spins $\ldots, \sigma_{-2}, \sigma_{-1}, \sigma_1, \sigma_2, \ldots$ and so can be written as

$$\langle \sigma_0 \rangle = \sum_{U:\, 0 \notin U} d(U)\, \sigma(U)$$

where the numbers $d(U)$ are given by

$$d(U) = \frac{1}{N} \sum_{\sigma} \sigma(U) \langle \sigma_0 \rangle$$

If c has finite support, these equations are well defined and $d(U)$ is nonzero for only finitely many U. Now

$$\frac{\partial f(V+t)}{\partial c(W)} = d((V+t) \triangle (W \backslash \{0\}))$$

where the symmetric difference $A \triangle B$ is defined to be $(A \cup B) \backslash (A \cap B)$. This implies

$$\frac{\partial F(V)}{\partial c(W)} = \sum_{t:\, V} d((V+t) \triangle (W \backslash \{0\}))$$

and so

$$\sum_{V:\,0\in V}\left|\frac{\partial F(V)}{\partial c(W)}\right| \leqslant \sum_{V:\,0\in V}\sum_{t:\,V}|d((V+t)\triangle(W\setminus\{0\}))|$$

This inequality and the bound (3.3) on $\|DF(c)\|$ imply

$$\begin{aligned}\|DF(c)\| &\leqslant \sup_W \sum_{V:\,0\in V}\sum_{t:\,V}|d((V+t)\triangle(W\setminus\{0\}))|\\ &= \sup_W \sum_{U:\,0\notin U}|d(U\triangle(W\setminus\{0\}))|\\ &= \sum_{U:\,0\notin U}|d(U)| \end{aligned}\tag{3.5}$$

We have used the fact that the sum over $V: 0\in V$ and $t: V$ is in one-to-one correspondence with the sum over $U: 0\notin U$, the correspondence being $U=V+t$. Then we have used the fact that as U ranges over all finite subsets not containing 0, $U\triangle(W\setminus\{0\})$ ranges over the same collection of sets. If we think of $\langle\sigma_0\rangle$ as a function of the boundary spins, then the last sum in (3.5) is just the norm of this function. We have proved the following lemma.

Lemma 3.1. For c with finite support,

$$\|DF(c)\| \leqslant \sum_{U:\,0\notin U}|d(U)| = \|\langle\sigma_0\rangle\|$$

It is convenient to define

$$D(c) = \sum_{U:\,0\notin U}|d(U)| = \|\langle\sigma_0\rangle\| \tag{3.6}$$

In the following we will restrict our attention to the following open subset of the Banach space:

$$\begin{aligned}O = \{c\colon c = c_0+\delta \text{ for some } c_0 \text{ and } \delta\\ \text{such that } c_0 \text{ has finite support,}\\ \|\delta\| < \ln 2,\ D(c_0)+\varepsilon(\|\delta\|) < 1\}\end{aligned}$$

where

$$\varepsilon(x) = 2(e^x-1)/(2-e^x) \tag{3.7}$$

The expression for $\varepsilon(\|\delta\|)$ is such that $\varepsilon(\|\delta\|)\to 0$ as $\|\delta\|\to 0$. Note that O

includes all c with finite support such that $D(c) < 1$. We will show that these c are dense in O.

The next lemma will be useful in bounding $\|DF(c+\delta) - DF(c)\|$.

Lemma 3.2. Let g be any function of σ with $\|g\| < \infty$. If c has finite support and $D(c) < 1$, then

$$\langle g \rangle_c \leqslant \|g\| \tag{3.8}$$

If c_1, c_2 have finite support, $D(c_1) < 1$, $D(c_2) < 1$, and $\|c_1 - c_2\| < \ln 2$, then

$$\|\langle g \rangle_{c_1} - \langle g \rangle_{c_2}\| \leqslant \varepsilon(\|c_1 - c_2\|)\, \|g\| \tag{3.9}$$

and

$$|D(c_1) - D(c_2)| \leqslant \varepsilon(\|c_1 - c_2\|) \tag{3.10}$$

The set of c with finite support and $D(c) < 1$ is dense in O. Thus, the definition of $\langle g \rangle_c$ extends in a unique continuous way to all $c \in O$, and (3.8) holds for all $c \in O$ and (3.9) holds for all $c_1, c_2 \in O$ such that $\|c_1 - c_2\| < \ln 2$.

Proof. To prove (3.8), it suffices to show $\|\langle \sigma(W) \rangle_c\| \leqslant 1$. If $0 \notin W$, then $\langle \sigma(W) \rangle_c = \sigma(W)$ and $\|\langle \sigma(W) \rangle_c\|$ is just 1. If $0 \in W$, then

$$\langle \sigma(W) \rangle = \langle \sigma_0 \rangle\, \sigma(W \backslash \{0\}) = \sum_U d(U)\, \sigma(U \,\triangle\, (W \backslash \{0\})$$

and so $\|\langle \sigma(W) \rangle_c\| \leqslant D(c) \leqslant 1$.

To prove (3.9), it also suffices to consider the case where $g = \sigma(W)$. Let $c = c_2$ and $\delta = c_1 - c_2$. The quantity we must bound can then be written as

$$\begin{aligned} &\langle \sigma(W) \rangle_{c+\delta} - \langle \sigma(W) \rangle_c \\ &\quad = \frac{\langle \sigma(W) \exp(\delta) \rangle_c}{\langle \exp(\delta) \rangle_c} - \langle \sigma(W) \rangle_c \\ &\quad = \frac{[\langle \sigma(W) \rangle_c + \langle \sigma(W)[\exp(\delta) - 1] \rangle_c]}{[\langle \exp(\delta) - 1 \rangle_c + 1]} - \langle \sigma(W) \rangle_c \end{aligned} \tag{3.11}$$

We are working in a Banach algebra, so

$$\|\exp(\delta) - 1\| = \left\| \sum_{n=1}^{\infty} \frac{1}{n!} \delta^n \right\| \leqslant \exp(\|\delta\|) - 1$$

By (3.8) we then have

$$\|\langle \exp(\delta) - 1 \rangle_c\| \leqslant \|\exp(\delta) - 1\| \leqslant \exp(\|\delta\|) - 1$$

The right side is less than 1 because $\|\delta\| = \|c_1 - c_2\| < \ln 2$, so we can expand the denominator in (3.11) as follows:

$$[1 + \langle \exp(\delta) - 1 \rangle_c]^{-1} = \sum_{n=0}^{\infty} (-1)^n (\langle \exp(\delta) - 1 \rangle_c)^n$$

Thus (3.11) equals

$$\langle \sigma(W) \rangle_c \sum_{n=1}^{\infty} (-1)^n (\langle \exp(\delta) - 1 \rangle_c)^n$$
$$+ \langle \sigma(W)[\exp(\delta) - 1] \rangle_c \sum_{n=0}^{\infty} (-1)^n (\langle \exp(\delta) - 1 \rangle_c)^n$$

The norm of this expression is

$$\leqslant \sum_{n=1}^{\infty} [\exp(\|\delta\|) - 1]^n + [\exp(\|\delta\|) - 1] \sum_{n=0}^{\infty} [\exp(\|\delta\|) - 1]^n$$
$$= \varepsilon(\|\delta\|)$$

The bound (3.10) follows from (3.9) because

$$|D(c_1) - D(c_2)| = | \, \|\langle \sigma_0 \rangle_{c_1}\| - \|\langle \sigma_0 \rangle_{c_2}\| \, | \leqslant \|\langle \sigma_0 \rangle_{c_1} - \langle \sigma_0 \rangle_{c_2}\|$$

To show that the c's with finite support and $D(c) < 1$ are dense in O, let $c \in O$. Then $c = c_0 + \delta$, where c_0 has finite support and $D(c_0) + \varepsilon(\|\delta\|) < 1$. Let δ_n have finite support with $\|\delta_n - \delta\| \to 0$. Then by (3.10), $D(c_0 + \delta_n) \leqslant D(c_0) + \varepsilon(\|\delta_n\|)$. Since $\varepsilon(\|\delta_n\|) \to \varepsilon(\|\delta\|)$, $D(c_0 + \delta_n) < 1$ for sufficiently large n. Thus, $c_0 + \delta_n \in O$ for large n and $c_0 + \delta_n \to c$.

The bound (3.9) and the density of the c's with finite support and $D(c) < 1$ imply that $\langle g \rangle$ has a unique continuous extension to all of O. Moreover, (3.8) and (3.9) hold on O. QED

Lemma 3.3. For c_1, c_2 with finite support and $D(c_1) < 1$, $D(c_2) < 1$, $\|c_1 - c_2\| < \ln 2$,

$$\|DF(c_1) - DF(c_2)\| \leqslant \varepsilon(\|c_1 - c_2\|) \tag{3.12}$$

$$\|F(c_1) - F(c_2)\| \leqslant \|c_1 - c_2\| \int_0^1 dt \, [1 + \varepsilon(t \, \|c_1 - c_2\|)] \tag{3.13}$$

Thus, $F(c)$ has a unique continuous extension to O, and this extension is differentiable and satisfies (3.12) and (3.13).

Proof:

$$\sum_V \left| \frac{\partial F(V, c_1)}{\partial c(W)} - \frac{\partial F(V, c_2)}{\partial c(W)} \right|$$

$$\begin{aligned}
&\leqslant \sum_{V,t:\,V} |\langle \sigma_0 \rangle ((V+t) \triangle (W \setminus \{0\}), c_1) \\
&\quad - \langle \sigma_0 \rangle ((V+t) \triangle (W \setminus \{0\}), c_2)| \\
&= \sum_U |\langle \sigma_0 \rangle (U, c_1) - \langle \sigma_0 \rangle (U, c_2)| \\
&= \| \langle \sigma_0 \rangle_{c_1} - \langle \sigma_0 \rangle_{c_2} \| \\
&\leqslant \varepsilon(\|c_1 - c_2\|)
\end{aligned}$$

The norm $\|DF(c_1) - DF(c_2)\|$ is bounded by the supremum over W of the above, so (3.12) follows.

The second inequality follows from

$$\begin{aligned}
F(c_1) - F(c_2) &= \int_0^1 dt \frac{d}{dt} F(c_2 + t(c_1 - c_2)) \\
&= \int_0^1 dt\, DF(c_2 + t(c_1 - c_2)) \cdot (c_1 - c_2)
\end{aligned}$$

We can then bound $\|DF(c_2 + t(c_1 - c_2))\|$ by $\|DF(c_2)\| + \varepsilon(t \, \|c_1 - c_2\|)$. The extension to all of O is straightforward. QED

We can now state and prove the main result of this section.

Theorem 3.4. Let $D(c, \beta_0)$ be given by Eq. (3.6) with $\beta = \beta_0$ and $\varepsilon(r)$ by Eq. (3.7). If there exists an approximate fixed point c_0 for $F(c, \beta_0)$ such that

$$\min_r r^{-1} \frac{\|F(c_0, \beta_0) - c_0\|}{1 - D(c_0, \beta_0) - \varepsilon(r)} < 1 \tag{3.14}$$

then there exists $\varepsilon > 0$ such that for $|\beta - \beta_0| < \varepsilon$, $F(c, \beta)$ has a fixed point $c(\beta)$. [It is understood that the minimum in (3.14) is only over $r > 0$ such that the denominator is positive.] Moreover, this family $c(\beta)$ of fixed points is differentiable in β, and the free energy $f(\varnothing, c(\beta), \beta)$ is analytic in β for $|\beta - \beta_0| < \varepsilon$. ($\beta_0$ can be complex and, by $|\beta - \beta_0| < \varepsilon$ we mean an open disk in the complex plane.)

Proof. We first show there is a fixed point for $\beta = \beta_0$. A standard argument for showing that there is a fixed point is to show that there is an approximate fixed point c_0, a positive number r, and a positive number ρ less than 1 such that

$$\|DF(c, \beta_0)\| \leqslant \rho \qquad \text{for} \quad \|c - c_0\| \leqslant r$$

and

$$\frac{\|F(c_0, \beta_0) - c_0\|}{1 - \rho} < r \tag{3.15}$$

Lemma 3.1 and the bound (3.12) imply that

$$\|DF(c_0 + \delta, \beta_0)\| \leqslant D(c_0, \beta_0) + \varepsilon(\|\delta\|)$$

Thus (3.14) implies (3.15). For β near β_0 we use the same approximate fixed point c_0. The quantities $D(c_0, \beta)$ and $\|F(c_0, \beta) - c_0\|$ are both continuous in β. Thus there is an $\varepsilon > 0$ such that (3.14) holds for $|\beta - \beta_0| < \varepsilon$, and so there is a fixed point for these values of β.

Let $c'(\beta)$ denote

$$(1 - DF(c, \beta))^{-1} \frac{\partial F}{\partial \beta}(c(\beta), \beta)$$

We show that $dc(\beta)/d\beta$ exists as a Frechet derivative and equals $c'(\beta)$. If $\Delta\beta$ is a small real number, then $c(\beta) + \Delta\beta c'(\beta)$ should be a good approximation to the solution $c(\beta + \Delta\beta)$ of the fixed-point equation $F(c(\beta + \Delta\beta), \beta + \Delta\beta) = c(\beta + \Delta\beta)$. Indeed, the techniques used above show that

$$\Big\| F(c(\beta) + \Delta\beta c'(\beta), \beta + \Delta\beta) - F(c(\beta), \beta) \\ - \Delta\beta DF(c(\beta), \beta)\, c'(\beta) - \Delta\beta \frac{\partial F}{\partial \beta}(c(\beta), \beta) \Big\|$$

is of order $(\Delta\beta)^2$. Using

$$F(c(\beta), \beta) = c(\beta) \qquad \text{and} \qquad DF(c(\beta), \beta)\, c'(\beta) + \frac{\partial F}{\partial \beta}(c(\beta), \beta) = c'(\beta)$$

this implies

$$\|F(c(\beta) + \Delta\beta c'(\beta), \beta + \Delta\beta) - c(\beta) - \Delta\beta c'(\beta)\|$$

is of order $(\Delta\beta)^2$. It follows that $\|c(\beta + \Delta\beta) - c(\beta) - \Delta\beta c'(\beta)\|$ is of order $(\Delta\beta)^2$. Thus, $dc/d\beta$ exists and equals $c'(\beta)$. Recall from the end of Section 2 that the free energy per site is given by $f(\varnothing, c(\beta), \beta)$. It now follows that the derivative of the free energy with respect to β exists and equals

$$\frac{df}{d\beta}(\varnothing, c(\beta), \beta) = \sum_{V:\, 0 \in V} \frac{\partial F}{\partial c(V)}(\varnothing, c(\beta), \beta)\, c'(V, \beta) + \frac{\partial f}{\partial \beta}(\varnothing, c(\beta), \beta) \quad \text{QED}$$

Remark. The above theorem does not address the question of the uniqueness of the fixed point. This appears to be a difficult question. We can, however, consider the following form of uniqueness. Suppose that (3.14) holds for all $\beta < \beta_0$. When $\beta = 0$, $c = 0$ is trivially a fixed point. Can we show that there is a unique continuous curve $c(\beta)$, $\beta < \beta_0$, of fixed points with $c(0) = 0$? The answer is yes if we make some additional assumptions that can be checked numerically. The proof of the theorem shows that $F(c, \beta)$ is a contraction in a neighborhood of $\{c(\beta)\colon |\beta - \beta_0| < \varepsilon\}$. Thus $F(c, \beta)$ cannot have two fixed points in this neighborhood. So we have a form of local uniqueness. Suppose that (3.14) holds for β_1, c_1 and for β_2, c_2. Let $\varepsilon_1, \varepsilon_2$ be the resulting ε's and suppose the regions $B_1 = \{\beta\colon |\beta - \beta_1| < \varepsilon_1\}$ and $B_2 = \{\beta\colon |\beta - \beta_2| < \varepsilon_2\}$ overlap. The theorem implies that there are two continuous functions $c_1(\beta)$ on B_1 and $c_2(\beta)$ on B_2 which are fixed points. We cannot conclude, however, that they agree on the overlap of B_1 and B_2. Suppose the overlap contains values of β at which they agree and at which they do not agree. The continuity then implies that we can find values of β in the overlap at which they do not agree but $\|c_1(\beta) - c_2(\beta)\|$ is as small as we like. Since $F(c, \beta)$ is a contraction in a neighborhood of the overlap, it can have at most one fixed point in this neighborhood, and so we have a contradiction. Thus, if $c_1(\beta)$ and $c_2(\beta)$ agree at one point in the overlap, then they must agree everywhere in the overlap. The contraction property of $F(c, \beta)$ implies that to show that two fixed points are equal we need only show that they are sufficiently close. This can be checked numerically since we can bound the distance from these fixed points to the approximate fixed points c_1 and c_2 and can compute $\|c_1 - c_2\|$. Thus, with extensive numerical checking we could prove that there is a differentiable curve $c(\beta)$ of fixed points for $\beta < \beta_0$ with $c(0) = 0$. (We have not attempted these numerical calculations.) Finally, we note that the above argument shows that there can be at most one such continuous curve with $c(0) = 0$.

We now turn to the problem of verifying (3.15). For small β we can try $c_0 = 0$ as the approximate fixed point. To check the above condition, we need a bound on $D(c)$. Until now we have been suppressing the β dependence of everything. We will now make it explicit by using a subscript, e.g., $F_\beta(c)$. Note that $F_\beta(c) = F_0(c + \beta\sigma_0\sigma_1 + \beta\sigma_0\sigma_{-1})$. Thus

$$D_\beta(0) = D_0(\beta\sigma_0\sigma_1 + \beta\sigma_0\sigma_{-1}) \leqslant D_0(0) + \varepsilon(2\beta)$$

by (3.12). Of course, $D_0(0)=0$, so

$$\|DF_\beta(0)\| \leqslant D_\beta(0) \leqslant \varepsilon(2\beta) = 2(e^{2\beta}-1)/(2-e^{2\beta})$$

Also,

$$\begin{aligned}\|F_\beta(0)-0\| &= \|F_0(\beta\sigma_0\sigma_1+\beta\sigma_0\sigma_{-1})\|\\ &\leqslant \int_0^1 dt\ \|DF_0(t\beta\sigma_0\sigma_1+t\beta\sigma_0\sigma_{-1})\|\ 2\beta\\ &\leqslant 2\beta\int_0^1 dt\ 2(e^{2\beta t}-1)/(2-e^{2\beta t})\end{aligned}$$

After a little calculation we find that $c_0=0$ works for $\beta \leqslant 0.21\beta_c$. [$\beta_c=\frac{1}{2}\log(1+\sqrt{2})=0.4407$.]

For larger values of β we use a computer to find a better approximate fixed point. Our approximate fixed point will have support in the set $\{-8, -7, -6,..., 3, 4, 5\}$. If c_0 is supported on this set, then $F(c_0)$ need not be. Writing $F(c_0)=\sum_V c(V)\,\sigma(V)$, we let $F^T(c_0)$ denote the sum over only those V with support in the above set. The finite-dimensional fixed-point equation $F^T(c_0)=c_0$ is then solved by iteration. The resulting c_0 is used as our approximate fixed point. We compute $\|F(c_0)-c_0\|$ and $\|\langle\sigma_0\rangle\|$ and then check condition (3.14). These numerical results are shown in Table I. We should point out that we do not use interval arithmetic in these calculations. For $\beta \leqslant 0.34 = 0.77\beta_c$ we see that the hypothesis of Theorem 3.4 is satisfied, and so we can conclude that there is a fixed point.

In Table I we have included some values of β for which we cannot verify (3.14). For some of these β the norm of DF at the approximate fixed point is greater than 1. Of course this does imply that the fixed-point equation does not have a solution or that F is not a contraction near this fixed point. It does imply that the methods of this section will not work for all β less than β_c. In the next section we will modify the fixed-point equation in an attempt to rectify this situation. We should remark that for these values of β with $\|DF(c_0)\|>1$, we find numerically that the approximate fixed-point equation does have a solution and is a contraction near this fixed point.

4. BETTER FIXED-POINT EQUATIONS

The numerical computations in Table I show that the simple norm of the previous section cannot be used to prove the existence of a solution to the fixed-point equation for β arbitrarily close to β_c. We have not been able to find another norm in which we can bound $\|DF(c)\|$. In this section

Table I. Test of Hypothesis (3.14) of Theorem 3.4 for Different Values of β[a]

β	$\|F(c)-c\|$	$\|DF(c)\|$	(3.14)
0.30	0.000403	0.712870	0.043229
0.31	0.000574	0.745147	0.077344
0.32	0.000815	0.778952	0.144278
0.33	0.001155	0.814541	0.286968
0.34	0.001633	0.852217	0.630897
0.35	0.002302	0.892331	1.652249
0.36	0.003230	0.935306	6.327473
0.37	0.004521	0.981624	
0.38	0.006298	1.031845	
0.39	0.008739	1.086598	
0.40	0.012065	1.146563	

[a] The approximate fixed point c is computed using the sites $-8,\ldots,5$. The last column is the left side of (3.14). If it is less than 1, then Theorem 3.4 implies there is an exact solution of the fixed-point equation.

we take a different approach. We will sum several spins at once rather than just a single spin. This leads to a new fixed-point equation for which there is a family of weighted l^1 norms that can be used. For example, we could sum out a 2 by 2 block as shown in Fig. 3. We denote the block of sites being summed out by B. For Fig. 3, $B=\{0, 1, 2, 3\}$. The generalization of (3.2) is

$$f(W)=N^{-1}\sum_{\sigma}\sigma(W)\ln\left\{\sum_{\sigma|_B}\exp\left[\sum_{V:\, V\cap B\neq\varnothing}c(V)\,\sigma(V)+H(\sigma)\right]\right\} \tag{4.1}$$

The sum over $\sigma|_B$ is over all spin configurations on the block B. (For Fig. 3 it is a sum over σ_0, σ_1, σ_2, σ_3.) The sum over σ is over all spin configurations on the sites outside the block B. Here $H(\sigma)$ is the appropriate generalization of the expression $\beta\sigma_0\sigma_1+\beta\sigma_0\sigma_{-1}$ in Eq. (3.2). It contains the terms in the Hamiltonian whose support intersects B but whose support does not contain any sites that have already been summed out. In Fig. 3 we have indicated the eight terms which are included in $H(\sigma)$ by drawing in the corresponding bonds. Of course, (4.1) is meaningful only if c has finite support.

The fixed-point equation will now involve shifts with respect to the block B rather than a single site. The appropriate generalization of (2.11) is

$$F(W, c, \beta)=\sum_{t:\, W}f(W+t, c, \beta)$$

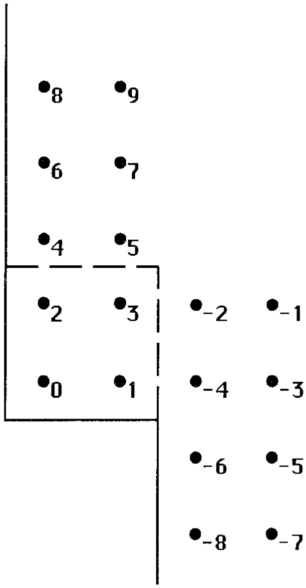

Fig. 3. An example of the kink for the fixed-point equations of Section 4. The spins at sites 0, 1, 2, and 3 will be summed over next.

where t is summed over all shifts by multiples of B such that (i) $B \cap (W+t) = \varnothing$, and (ii) $W+t$ has the same geometric shape as W.

The fixed point equation is again given by (2.12). For Fig. 3 we have labeled the sites so that the shifts with respect to the block B are given by adding or subtracting a multiple of 4 to the site label. Thus, the sum over t in the above is just the sum over integer multiples of 4 subject to (i) and (ii).

The advantage of summing out several spins at once is that we can now introduce several parameters in our norm and optimize our estimates with respect to these parameters. For each site $i \in B$ let $\mu_i \geqslant 0$. We then define μ_i for all sites i by the requirement that μ_i be invariant under translations with respect to B. For Fig. 3 this means that $\mu_{i+4n} = \mu_i$. The norm is defined as follows:

$$\begin{aligned} \|c\| &= \sum_W |c(W)|\, e^{\mu(W)} \\ \mu(W) &= \sum_{i \in W} \mu_i \end{aligned} \tag{4.2}$$

The restriction that $\mu_i \geqslant 0$ implies that we have a Banach algebra as follows:

$$\begin{aligned}\|cd\| &= \sum_W |(cd)(W)|\, e^{\mu(W)} \\ &= \sum_W e^{\mu(W)} \left| \sum_{X, Y:\, X \triangle Y = W} c(X)\, d(Y) \right|\end{aligned}$$

Each $\mu_i \geqslant 0$, so we have $\mu(W) = \mu(X \triangle Y) = \mu(X) + \mu(Y) - \mu(X \cap Y) \leqslant \mu(X) + \mu(Y)$. Thus, the above is bounded by $\|c\| \cdot \|d\|$. QED

For a function g of σ we define

$$\langle g \rangle = Z^{-1} \sum_{\sigma | B} g(\sigma) \exp\left[\sum_{V:\, V \cap B \neq \varnothing} c(V)\, \sigma(V) + H(\sigma) \right] \tag{4.3}$$

where Z is defined by $\langle 1 \rangle = 1$. This expectation depends on c, and when we need to make this dependence explicit, we will write $\langle g \rangle_c$. This expectation is a function of the σ_i with $i \notin B$, so there are numbers $\langle g \rangle(W)$ for each set W outside the block such that

$$\langle g \rangle = \sum_{W:\, W \cap B = \varnothing} \langle g \rangle(W)\, \sigma(W)$$

We then define

$$\|\langle g \rangle\| = \sum_{W:\, W \cap B = \varnothing} |\langle g \rangle(W)|\, e^{\mu(W)}$$

We then have the following bound on the norm of the Jacobian.

Lemma 4.1.

$$\|DF(c)\| \leqslant \max_{A:\, A \subset B,\, A \neq \varnothing} e^{-\mu(A)}\, \|\langle \sigma(A) \rangle\| \tag{4.4}$$

Proof. With our weighted l^1 norm, the norm of $DF(c)$ is easily shown to be bounded by

$$\|DF(c)\| \leqslant \sup_W e^{-\mu(W)} \sum_V \left| \frac{\partial F(V)}{\partial c(W)} \right| e^{\mu(V)} \tag{4.5}$$

The sup over W and the sum over V are over all sets which contain at least one site in the block B and any number of sites outside of B (possibly none). We decompose W into the sites inside B and the sites outside B: $W = A \cup D$, $A = W \cap B$, $D = W \setminus B$. Then

$$\begin{aligned}\frac{\partial F(V)}{\partial c(W)} &= \sum_{t:\, V} \frac{\partial f(V+t)}{\partial c(W)} \\ &= \sum_{t:\, V} \langle \sigma(A)\, \sigma(D) \rangle (V+t)\end{aligned} \tag{4.6}$$

D is disjoint from the block B, so $\langle\sigma(A)\,\sigma(D)\rangle = \langle\sigma(A)\rangle\,\sigma(D)$. Hence

$$\langle\sigma(A)\,\sigma(D)\rangle(V+t) = \langle\sigma(A)\rangle(D \triangle (V+t))$$

Thus

$$e^{-\mu(W)} \sum_V \left|\frac{\partial F(V)}{\partial c(W)}\right| e^{\mu(V)}$$

$$\leqslant e^{-\mu(W)} \sum_V \sum_{t:\,V} |\langle\sigma(A)\rangle(D \triangle (V+t))|\, e^{\mu(V)} \tag{4.7}$$

Since $(V+t) \subset D \cup [D \triangle (V+t)]$, we have

$$\mu(V) = \mu(V+t) \leqslant \mu(D) + \mu(D \triangle (V+t))$$

This inequality and the trivial equality $\mu(W) = \mu(A) + \mu(D)$ imply that (4.7) is

$$\leqslant e^{-\mu(A)} \sum_V \sum_{t:\,V} |\langle\sigma(A)\rangle(D \triangle (V+t))|\, e^{\mu(D \triangle (V+t))} \tag{4.8}$$

In the sum over V and t: V, the set $D \triangle (V+t)$ runs over all subsets of the set of sites outside the block. So the above is equal to $e^{-\mu(A)}\,\|\langle\sigma(A)\rangle\|$. The lemma follows. QED

We now define

$$D(c) = \max_{A:\,A \subset B,\,A \neq \varnothing} e^{-\mu(A)}\,\|\langle\sigma(A)\rangle\| \tag{4.9}$$

The open set O is defined as in Section 3, except that we use the above definition of $D(c)$. Equations (4.1) and (4.3) only make sense for c with finite support. The following lemma extends the definitions to all of O.

Lemma 4.2. For any function g of σ with $\|g\| < \infty$ there is a unique continuous extension of $\langle g\rangle_c$ to all $c \in O$. For any $c \in O$,

$$\|\langle g\rangle_c\| \leqslant \|g\| \tag{4.10}$$

and if $c_1, c_2 \in O$ with $\|c_1 - c_2\| < \ln 2$, then

$$\|\langle g\rangle_{c_1} - \langle g\rangle_{c_2}\| \leqslant \varepsilon(\|c_1 - c_2\|)\,\|g\| \tag{4.11}$$

There is a unique continuous extension of $F(c)$ to all of O. This extension is differentiable and if $c_1, c_2 \in O$ with $\|c_1 - c_2\| < \ln 2$, then

$$\|DF(c_1) - DF(c_2)\| \leqslant \varepsilon(\|c_1 - c_2\|) \tag{4.12}$$

$$\|F(c_1) - F(c_2)\| \leqslant \|c_1 - c_2\| \int_0^1 dt\,[1 + \varepsilon(t\,\|c_1 - c_2\|)] \tag{4.13}$$

where $\varepsilon(x)$ is defined by (3.7).

Proofs. As in Section 3, we first prove the estimates for c's with finite support. The definitions of $\langle g \rangle_c$, $DF(c)$, and $F(c)$ and the above bounds then extend to all of O by continuity. To prove (4.10), it suffices to show $\|\langle \sigma(W) \rangle_c\| \leqslant e^{\mu(W)}$. As before, we let $W = A \cup D$ with $A = W \cap B$, $D = W \setminus B$. Then $\langle \sigma(W) \rangle_c = \langle \sigma(A) \rangle_c \sigma(D)$. So

$$\begin{aligned}\|\langle \sigma(W) \rangle_c\| &= \sum_V |\langle \sigma(W) \rangle_c (V)| \, e^{\mu(V)} \\ &= \sum_V |\langle \sigma(A) \rangle_c (V \triangle D)| \, e^{\mu(V)}\end{aligned}$$

Since $\mu(V) \leqslant \mu(V \triangle D) + \mu(D) = \mu(V \triangle D) + \mu(W) - \mu(A)$, this is

$$\begin{aligned}&\leqslant \sum_V |\langle \sigma(A) \rangle_c (V \triangle D)| \, e^{\mu(V \triangle D) + \mu(W) - \mu(A)} \\ &= e^{\mu(W) - \mu(A)} \, \|\langle \sigma(A) \rangle_c\| \leqslant e^{\mu(W)}\end{aligned}$$

where we have used the fact that as V runs over all sets disjoint from B, so does $V \triangle D$. The proofs of (4.11)–(4.13) are virtually identical to the proofs of (3.9), (3.12), and (3.13) in Lemmas 3.2 and 3.3. QED

Theorem 3.4 with $D(c)$ given by (4.9) also holds for these new fixed-point equations. In Table II we check hypothesis (3.14) for a few values of β. Finding the best choice of the weights μ_0, μ_1, μ_2, μ_3 is not trivial. The choices shown in Table II were made by searching for a minimum of $\|DF(c_0)\|$, and are probably not the best choices. The main point of Table II is to show that the freedom to choose the μ_i does indeed yield better bounds on $\|DF(c)\|$. Unfortunately, the 2 by 2 block involves more sites and hence more computation. In particular, for a fixed amount of computation the bound on $\|F(c_0) - c_0\|$ will be larger for the 2 by 2 block than for the single-site block. This is the reason that the largest β for which

Table II. Test of Hypothesis (3.14) of Theorem 3.4 Using the Weighted Norm Defined in (4.2)[a]

β	$\|F(c) - c\|$	$\|DF(c)\|$	(3.14)	μ_0	μ_1	μ_2	μ_3
0.34	0.009569	0.699036	0.937822	0.000	0.325	0.188	0.537
0.35	0.012706	0.756317	1.864607	0.000	0.288	0.163	0.475
0.36	0.016503	0.821021	4.393333	0.000	0.250	0.125	0.425

[a] The values of μ are given in the last four columns. The flexibility in the choice of μ leads to a smaller bound on $\|DF\|$. The approximate fixed point was computed using sites from -16 to 10 as labeled in Fig. 2.

we can verify the hypothesis of Theorem 3.4 with a single-site block is about the same as for the case of a 2 by 2 block. The important question, which we cannot answer, is whether or not for all $\beta < \beta_c$ one can find a block size, a weighted norm, and an approximate solution such that Theorem 3.4 allows us to conclude that the fixed-point equation has a solution.

5. CORRELATION FUNCTIONS

At first sight our approach appears to use translation invariance in such a strong way that there is no hope of defining the correlation functions in this approach. This is not the case, however, and in this section we show how to extract the correlation functions from the fixed-point equation.

Until now only the sites shown in Fig. 2 occurred in the definition of $F(c)$. We must now include sites to the right of those labeled in Fig. 2, so a slight change in notation is required. Sites will be labeled $i = (i_1, i_2)$, where i_1 is a row index and i_2 is a column index. The columns will have a kink in them, so that the sites in Fig. 2 will all have $i_2 = 0$.

To obtain the two-point function we add a small perturbation to the Hamiltonian,

$$H = \beta \sum_{\langle ij \rangle} \sigma_i \sigma_j + \alpha \sum_s \sigma_s \sigma_{k+s} \tag{5.1}$$

Here s is summed over all sites in the square lattice and α is small. The fixed-point equation for this new Hamiltonian is obtained by adding the term $\alpha\sigma_0\sigma_k$ to $\beta\sigma_0\sigma_1 + \beta\sigma_0\sigma_{-1}$ in Eq. (2.4). [We can assume without loss of generality that $k_2 \geq 0$, i.e., k is to the right of the line in Fig. 2. The subscripts 0, 1, and -1 should now be replaced by $(0, 0)$, $(1, 0)$, and $(-1, 0)$.] The fixed point $\sum_V c(V)\,\sigma(V)$ will now involve V's which are subsets of $\{(i_1, i_2): 0 \leqslant i_2 \leqslant k_2\}$.

Let c_0 be an approximate fixed point for $\beta = \beta_0$ for which the contraction mapping argument applies and shows that there is an exact fixed point. As $\alpha \to 0$, $\|F(c_0, \beta_0, \alpha) - c_0\| \to \|F(c_0, \beta_0, 0) - c_0\|$ and $\|DF(c, \beta_0, \alpha)\| \to \|DF(c, \beta_0, 0)\|$. Hence, for sufficiently small α the contraction mapping argument shows that the equation $F(c, \beta_0, \alpha) = c$ has an exact solution. Furthermore, the resulting free energy is an analytic function of β and α for β close to β_0 and α close to 0.

Formally, the derivative of the free energy per site with respect to α at $\alpha = 0$ equals

$$|\Lambda|^{-1} \sum_t \langle \sigma_t \sigma_{k+t} \rangle = \langle \sigma_0 \sigma_k \rangle$$

by translation invariance. Thus, we define the correlation function $\langle \sigma_0 \sigma_k \rangle$ to be the derivative of the free energy per site (as defined by the fixed-point equation) with respect to α at $\alpha = 0$. As we have just argued, this derivative exists and is an analytic function of β for any value of β for which the methods of Sections 3 and 4 show that the fixed-point equation has a solution.

Unfortunately, this simple argument cannot establish the exponential decay of the truncated correlation functions. To do this, we must introduce a site-dependent magnetic field, i.e., add a term $\sum_i h_i \sigma_i$ to the Hamiltonian. (Truncated correlation functions are obtained by differentiating the free energy with respect to some of the h_i and then setting all the h_i equal to 0.) The partition function for Fig. 2 will now depend on all the h_i to the left of the line as well as the spins σ. It appears that we have lost translational invariance in a serious way, and the arguments of Section 2 will no longer apply. This is not so. For example, after summing over σ_0, the partition function should be the same function of h_i that the partition function before summing over σ_0 was of $h_{i'}$, where i' is the translate of i by one lattice spacing. Thus, we have a fixed-point equation in which c is now a function not only of the spins σ_i, but also of the fields h_i to the left of the line in Fig. 2. To establish the existence of a solution to this more complicated equation requires introducing a norm on this new space of c's. This can be done, but we will not pursue it here.

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